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BRINGING A NON-LINEAR MANOEUVRING OBJECT TO THE OPTIMAL POSITION IN THE SHORTEST TIME†

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A different game problem with two players (cars), in which one player (car) pursues the other, is considered. The roles of the players are fixed, and the functional to be minimized (for player I) and maximized (for player II) is the maximum value of a given scalar non-negative function (the performance index) of the phase vector along the trajectory of the dynamical system over a fairly long time interval. A zero value of the performance index corresponds to the situation in which the pursuer is behind the evader at a given distance from it, and the velocity vectors are codirectional and lie on the same straight line. A detailed investigation is presented of the special case in which the car being pursued is at rest, and the pursuer is moving in the plane at a velocity of constant magnitude subject to a certain constraint on its turning radius. The game ends when the car is moving in a circle of given radius, in which case its velocity vector must point toward the centre of the circle. The relations of the Pontryagin maximum principle characterizing optimal open-loop controls are written out and analysed. The main result of the paper is the synthesis of an optimal feedback control.

To model the control of two or more moving objects under conflict conditions, when the manoeuvring objects have to achieve contrary goals and their possibilities are different, wide use has been made of the tools of differential game theory, which have seen considerable development over the past few decades [1, 2]. The exact solution of problems of differential game theory presents considerable difficulties, particularly for non-linear systems. In some cases, in order to devise algorithms for the numerical synthesis of controls, it proves useful to consider a simplified problem, whose solution can be completed, and then to use the results when proceeding to solve the initial problem.

In this paper we investigate the limiting case of a two player game problem of pursuit in a plane, which reduces the initial game problem to a non-linear optimal control problem.

1. DIFFERENT GAME

The game takes place in a plane. Two players participate: pursuer P (player I) and evader E (player II). The velocities of players P and E are constant and equal to V_1 and V_2; the radii of curvature of the trajectories of motion are bounded below by given quantities R_1 and R_2, respectively. It is assumed that both players, at each instant of time, control the choice of the curvature of their trajectories, using information about the actual state of the system in the phase space of coordinates $(r, \varphi_1, \varphi_2)$ and about the quantities V_1, V_2, R_1 and R_2. Here r is the distance between the players, and φ_1 (φ_2) is the angle between the velocity vectors of player I (II) and the segment PE. The angle φ_1 is measured counterclockwise and φ_2 clockwise (Fig. 1). Angles differing by $2\pi n$, where n is an integer, are identified. The quantities V_k and R_k are assumed to be non-negative; more precisely, $V_1 \geq 0$, $V_2 \geq 0$, $R_1 > 0$, $R_2 > 0$.

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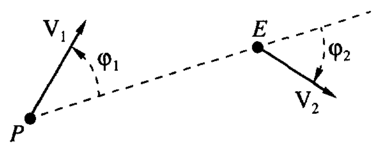

Fig. 1

An equivalent description of the dynamics of the points P and E comprises points possessing a certain mass that control their motion by means of a force perpendicular to their velocity and bounded in magnitude. Such a model is not infrequently used to describe the motion of aircraft.

Arguments analogous to those used previously in [1] show that the dynamics of the relative position of the players is described by a non-linear system of three differential equations

$$\begin{aligned} \dot{r} &= V_2\cos\varphi_2 - V_1\cos\varphi_1 \\ \dot{\varphi}_1 &= \frac{V_1\sin\varphi_1 + V_2\sin\varphi_2}{r} - \frac{V_1}{R_1}u, \quad \dot{\varphi}_2 = \frac{V_2}{R_2}\upsilon - \frac{V_1\sin\varphi_1 + V_2\sin\varphi_2}{r} \end{aligned} \tag{1.1}$$

and constraints

$$|u| \le 1, \quad |\upsilon| \le 1 \tag{1.2}$$

where u and υ are the control parameters of player P and E.

By the physical meaning of the problem, the variable r is non-negative, which is in fact a phase constraint for system (1.1). This constraint will be taken into consideration in constructing a synthesis.

The payoff in the game is the quantity

$$J[u, \upsilon] = \min_{t\ge 0} L(r(t), \varphi_1(t), \varphi_2(t)) \tag{1.3}$$

where $r(t)$, $\varphi_1(t)$, $\varphi_2(t)$ is the trajectory of system (1.1) corresponding to the given controls u and υ of the players, and the performance index L has the form

$$L(r, \varphi_1, \varphi_2) = A_1\varphi_1^2 + A_2\varphi_2^2 + A_3(r-R)^2 \tag{1.4}$$

where A_1, A_2, A_3 and R are given non-negative constants. The controls u and υ may be open-loop or in synthesis form; the main condition imposed upon them is that system (1.1) should be solvable. The function $L(r, \varphi_1, \varphi_2)$ may be treated as the probability that player II will be hit by player II from the position $(r, \varphi_1, \varphi_2)$. The lower the value of the function L at a given time, the more preferable player I's position for hitting player II. Hence player I will try to choose this strategy so as to minimize $J[u, \upsilon]$, while player II, on the contrary, will try to maximize it. Obviously, the most satisfactory time to finish the game from the standpoint of player I corresponds to the position $(R, 0, 0)$, which gives the performance index L a global minimum equal to zero. From the physical standpoint, this position corresponds to the situation in which the pursuer is behind the evader at a distance R from him, and the players' velocities are codirectional and lie on the same straight line.

We will consider the problem of finding the optimal strategy $u^*(r, \varphi_1, \varphi_2, t)$ of player I and the corresponding guaranteed value of the functional $J[u, \upsilon]$

$$L^* = \min_{|u|\le 1}\max_{|\upsilon|\le 1} J[u, \upsilon] = \min_{|u|\le 1}\max_{|\upsilon|\le 1}\min_{t\ge 0} L(r(t), \varphi_1(t), \varphi_2(t)) \tag{1.5}$$

The function L^* (the value of the game) depends on the initial position of the system $(r_0, \varphi_1^0, \varphi_2^0)$. It will suffice to consider maximization with respect to υ in (1.5) for open-loop controls $\upsilon(t)$, $|\upsilon(t)| \le 1$. A more detailed formulation of the game problem is not given, since a simplified version will be considered below.

Let us determine the number of essential parameters of system (1.1) on the assumption that $R_1 > 0$, $V_2 > 0$. We make the following replacement of variables

$$r' = \frac{r}{R_1}, \quad \lambda = \frac{V_2}{V_1}, \quad V' = \frac{V}{V_1}, \quad \mu = \frac{V_2/R_2}{V_1/R_1}, \quad \lambda \ge 0, \quad \mu \ge 0 \tag{1.6}$$

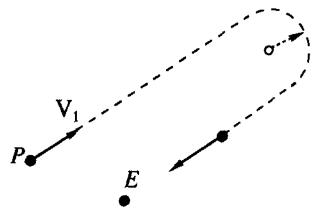

Fig. 2

Then system (1.1) takes the form

$$\dot{r} = \lambda\cos\varphi_2 - \cos\varphi_1$$
$$\dot{\varphi}_1 = \frac{\sin\varphi_1 + \lambda\sin\varphi_2}{r} - u, \quad \dot{\varphi}_2 = \mu v - \frac{\sin\varphi_1 + \lambda\sin\varphi_2}{r} \tag{1.7}$$

with two essential constants λ and μ instead of the four in system (1.1). the constraints (1.2) are retained. The quantity R in function (1.4) becomes $R' = R/R_1$.

The game problem we have formulated, with a functional of the minimum type, is one of the most difficult problems of differential game theory, since the Bellman optimality principle is not satisfied for it, and the corresponding dynamic programming problem reduces to a Bellman equation with an unknown boundary.

In this paper the problem will be considered with limiting values of the parameters, so that it will be possible to investigate the special case in which the velocity of the evader is significantly less than that of the pursuer ($V_2 \ll V_1$).

Let us assume that the following three parameters in relations (1.4) and (1.7) vanish

$$\lambda = 0, \quad \mu = 0, \quad A_2 = 0 \tag{1.8}$$

In physical terms, this limiting case means that the evader is at rest, and his control vanishes identically over the interval of motion. It can easily be shown that with these simplifications an infinite set of strategies $u^*(t)$ of the pursuer exists that will steer the system to the position $(R, 0, 0)$ best for the latter player. Indeed, from any initial position, the pursuer must move away a sufficient distance from the evader, change direction in such a way that $\varphi_1 = 0$, $\varphi_2 = 0$, and then proceed to the point indicated along a straight line (see Fig. 2).

Note that on taking the limit (1.8) the game problem reduces to an optimal control problem for one player. When that is done, since an infinite set of strategies $u^*(t)$ exists (in the sense of (1.5)) that make the functional L a global minimum equal to zero, it seems reasonable to consider a new, additional functional and look for a strategy that is optimal in the sense of the new functional. We shall consider as the additional functional the time needed to reach the point$(R, 0, 0)$.

2. FORMULATION OF THE OPTIMAL CONTROL PROBLEM

Suppose the motion of a non-linear controlled object in a two-dimensional phase space is described for $t \ge t_0$ by the following system of equations

$$\dot{x}_1 = -\cos x_2, \quad \dot{x}_2 = -u + \frac{\sin x_2}{x_1} \tag{2.1}$$

where x_1 and x_2 are the coordinates of the phase vector of system (2.1), and u is a control parameter satisfying the constraint

$$|u| \le 1 \tag{2.2}$$

It is assumed that the state of the object at the initial time is given:

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0 \tag{2.3}$$

System (2.1), (2.2) describes the differential game (1.7), (1.2) with the simplifications (1.8), when the game degenerates to an optimal control problem. Then x_1 is the distance r between the players, and x_1 is the angle φ_1 (Fig. 1). Note that if $x_1 = 0$, the variable x_2 becomes meaningless, and system (2.1) has a singularity which disappears when other variables are used, such as Cartesian coordinates in absolute space (see [6]). Below, when constructing an optimal synthesis, such variables will also be used.

The class of admissible controls will be the set of piecewise-continuous scalar functions $u(t)$ satisfying constraint (2.2).

It is required to find a time-optimal control u^* that will steer the system from some initial state (2.3) to the target point $O(R, 0)$

$$x_1(t_1) = R, \quad x_2(t_1) = 0 \tag{2.4}$$

and the corresponding trajectory and time $T = t_1 - t_0$.

3. SYNTHESIS OF THE CONTROL

In this section a certain synthesis of the control, whose optimality will be proved later, will be proposed.

We will first carry out some auxiliary constructions using the Pontryagin Maximum Principle [3, 4], and then describe the proposed synthesis.

Let ψ_1 and ψ_2 denote the variables conjugate to x_1 and x_2. The Hamiltonian of system (2.1) is

$$H(t, \psi, x, u) = -\psi_1 \cos x_2 - \psi_2 u + \psi_2 \frac{\sin x_2}{x_1} \tag{3.1}$$

By maximizing the Hamiltonian subject to the constraint (2.2), we deduce that the optimal control u^* has the form

$$u^*(t) = \begin{cases} 1, & \psi_2 < 0 \\ -1, & \psi_2 > 0 \\ \theta, & \text{where} \quad \theta \in [-1, 1], \quad \psi_2 = 0 \end{cases} \tag{3.2}$$

Let us determine what values the quantity θ may take. The canonical equations of the maximum principle may be written in the form

$$\begin{gathered} \dot{x}_1 = -\cos x_2 = f_1(x_2), \quad \dot{x}_2 = -u + \frac{\sin x_2}{x_1} = f_2(x_1, x_2, u) \\ \dot{\psi}_1 = \psi_2 \frac{\sin x_2}{x_1^2}, \quad \dot{\psi}_2 = -\psi_1 \sin x_2 - \psi_2 \frac{\cos x_2}{x_1} \end{gathered} \tag{3.3}$$

If $\psi_2 \equiv 0$, then $\dot{\psi}_2 \equiv 0$, and it follows from the last equation of (3.3) that on such a trajectory $\psi_1 \sin(x_2) \equiv 0$. Since the maximum principle requires that ψ_1 and ψ_2 should not vanish simultaneously, it follows from the continuity of the function $x_2(t)$ in the interval $[t_0, t_1]$ that $x_2(t) \equiv \pi n$, where n is an integer. Then $\dot{x}_2(t) \equiv 0$ and it follows from the second equation of (3.3) that if $\psi_2 \equiv 0$ then $u \equiv 0$. This means that θ may take the single value zero. The corresponding trajectories are conveniently interpreted in the original space. For example, if player P uses the constant control $u = 1$ over some time interval, then the point P will move clockwise along the unit circle; if $u = -1$, it will move counterclockwise, and if $u = 0$, it will move along a straight line.

Thus, the optimal trajectories in absolute space must consist of arcs of circles of unit radius and of straight lines. Moreover, if the point is moving along a straight line, the phase coordinate x_2 is equal to $n\pi$, where n is an integer, which implies motion of player P along the line connecting the players. An analogous class of trajectories was used in [5] to construct attainability regions.

Since the functions f_1 and f_2 in Eqs (3.3) are 2π-periodic in x_2, the phase portrait of the optimal trajectories must be 2π-periodic in that variable.

We also note that by symmetry and the properties of the functions f_1 and f_2

$$f_1(2\pi - x_2) = f_1(x_2), \quad f_2(x_1, 2\pi - x_2, -u) = -f_2(x_1, x_2, u)$$

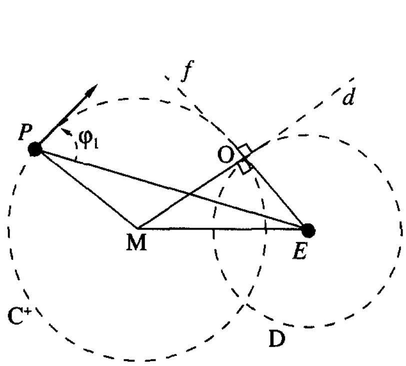

Fig. 3

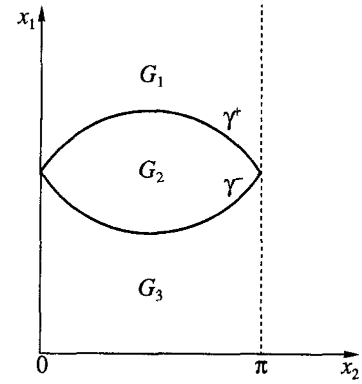

Fig. 4

it follows that the optimal trajectories considered in the interval $[0, 2\pi]$ are symmetrical about the straight line $x_2 = \pi$, which, as will be shown later, is a singular surface in the terminology of [1]. It will therefore suffice to synthesize a solution in the region

$$G = \{(x_1, x_2) : 0 \le x_2 \le \pi,\ x_1 \ge 0\}$$

Denote the half-lines $\{x_2 = 0, x_1 \ge 0\}$, $\{x_2 = \pi, x_1 \ge 0\}$ by m and l, respectively. It follows from the above arguments that all straight sections of the optimal trajectories in the region G belong either to m or to l.

Let C^+ and C^- denote circles of unit radius in absolute space, which are trajectories of the motion of player P when the latter, beginning at time t_0, applies the constant control $u = 1$ and $u = -1$, respectively. Let us find the set of initial positions (2.3) of player P from which the player may reach the point O moving along a circle C^+ or C^-.

To fix our ideas, suppose player P, applying the constant control $u = 1$ from some initial position (2.3), reaches point O at time t_1. Then the circles C^+ and D (with centre at the point E and radius R) intersect at O, where they have perpendicular tangents f and d, respectively (Fig. 3). It follows that

$$|ME|^2 = |MO|^2 + |OE|^2 = 1 + R^2$$

Consider the triangle PEM for which

$$|PE| = x_1, \quad |ME|^2 = 1 + R^2, \quad |MP| = 1, \quad \angle MPE = \pi/2 - x_2$$

By the cosine theorem,

$$x_1^2 - 2x_1 \sin x_2 - R^2 = 0$$

whence it follows that

$$x_1(x_2) = x_1^+(x_2, R) \tag{3.4}$$

Here and henceforth, we are using the notation

$$x_1^+(x_2, R) = \sin x_2 + \sqrt{\sin^2 x_2 + R^2}, \quad x_1^-(x_2, R) = -\sin x_2 + \sqrt{\sin^2 x_2 + R^2}$$

Reasoning similarly for the case $u = -1$, we obtain

$$x_1(x_2) = x_1^-(x_2, R) \tag{3.5}$$

Denote the curves defined by Eqs (3.4) and (3.5) by γ^+ and γ^-, respectively. They divide the set $G' = G\backslash(m \cup l)$ into three subsets (Fig. 4)

$$G_1 : x_1 \ge x_1^+(x_2, R), \quad G_2 : x_1^-(x_2, R) \le x_1 < x_1^+(x_2, R), \quad G_3 : x_1 < x_1^-(x_2, R)$$

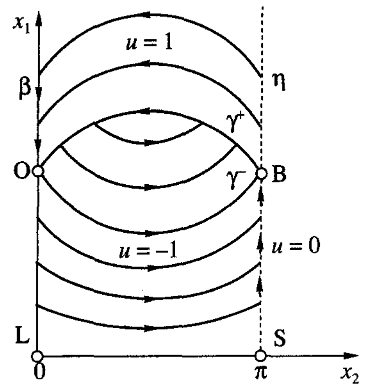

Fig. 5

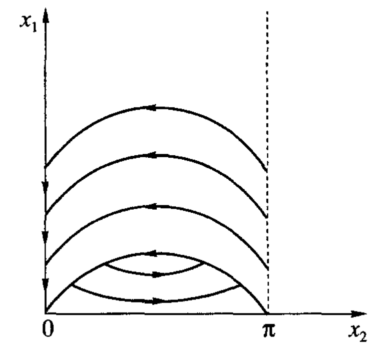

Fig. 6

The proposed synthesis will be described in the set G. In the constructions we shall use trajectories of the point P in absolute space, but the phase portrait will be given in coordinates (x_1, x_2).

By construction, the points of the set G_1 will have the property that a trajectory exists consisting of an arc of the circle C^+ and a segment of the half-line m that takes player P to the target point. We will therefore assume in the proposed synthesis that in the region G_1 we have $u = 1$.

For initial points in the set G_2, the trajectories described above do not enable us to bring player P to the target point, because of the proximity of the circles C^+ and D (Fig. 3). Here success will be achieved by using the control $u = -1$ until the curve γ^+ is reached, and there the control is already known. In the region G_2, therefore, we put $u = -1$.

The situation for the set G_3 is more complicated. Because the players are close together here, one must first increase the distance between them. One possible manoeuvre guaranteeing the rapid motion of player P away from player E will consist of an arc of the circle C^+ and a segment of the half-time l. Moving in a straight line, player P will, after some time, reach the curve γ^+, along which he will then reach the target point O. With that manoeuvre, $u = -1$ in the set G_3.

Consider the points $L(0, 0)$, $O(R, 0)$, $S(0, \pi)$, $B(R, \pi)$. The points L and O (S and B) divide the half-line m (the half-line l) into two parts: a segment LO (segment SB) and a half-line β (half-line η). We put $u = 0$ if $P \in SB \cup \beta$. But if $P \in LO \cup \eta$, there are two possibilities: $u = 1$ or $u = -1$. In the terminology of [1], the segment SB and half-line β are universal surfaces, while LO and the half-line η are dispersal surfaces.

The synthesis thus obtained is illustrated in Fig. 5.

The maximum principle was used in [6, 7] to solve a similar time-optimal problem, that of steering system (2.1), (2.2), but in different phase variables, from the initial position (2.3) to the origin $x_1(t_1) = 0$, $x_2(t_1) = 0$; the optimal synthesis obtained, expressed in coordinates (x_1, x_2), has the form shown in Fig. 6; it is identical with the synthesis proposed here in the case when $R = 0$ in (2.4). Similar problems were considered in [8, 9]. Some elements of the synthesis proposed here were constructed in [1].

4. PROOF OF THE OPTIMALITY OF THE SYNTHESIS

We shall show that the synthesis constructed in Section 3 satisfies the necessary optimality condition of the non-linear problem – the Pontryagin Maximum Principle.

Let $q \in G$ be an arbitrary but fixed point, and consider the process corresponding to the synthesis, steering player P from that point to the target point O at time t_q. To prove the validity of the maximum principle, it will suffice to exhibit a vector-valued function $\psi(t)$, defined in the interval $[t_0, t_q]$, such that (3.3) holds in that interval and the following two conditions hold there

$$\psi(t) \neq \bar{0} \tag{4.1}$$

$$u = \arg\max_{|u| \le 1} H(t, \psi(t), x(t), u(t)) = -\operatorname{sign}(\psi_2(t)) \tag{4.2}$$

By suitable normalization of the vector of conjugate coordinates, we can obtain from (3.1)

$$H(t, \psi(t), x(t), u(t)) = 1, \quad t \in [t_0, t_q] \tag{4.3}$$

Let us find a value of the function ψ in the time interval over which $u = 0$ (corresponding to the motion of player P along either the half-time β or the segment SB). It follows from the previous discussion that in that case $\psi_2 = 0$. Then, taking into account that the Hamiltonian is constant, we deduce that for $u = 0$

$$-\psi_1 \cos x_2 = 1 \tag{4.4}$$

It follows from this equality that $\psi_1 = -1$ if $P \in \beta$, and $\psi_1 = 1$ if $P \in SB$.

The point q in G is either in one of the sets G_1, G_2, G_3, or $q \in \beta \cup SB$. The segment LO and half-line η need not be considered, since they do not contain trajectories.

Let $q \in G_3$. Then the trajectory is the union of an arc of the circle C^-, a segment of the half-line l and an arc of the circle corresponding to the curve γ^+. Let N denote the point at which the trajectory of motion switches from the circle C^- to the half-line l. Assume that player P reaches the point N at a time t_N, $t_0 < t_N < t_q$, and the point B at a time t_B, $t_N < t_B < t_q$. As shown previously, the value of the function ψ in the interval $[t_N, t_B]$ is known

$$\psi(t) = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix}, \quad \text{if} \quad t \in [t_N, t_B] \tag{4.5}$$

We will show that the function $\psi(t)$, defined in $[t_B, t_q]$ as the solution of the Cauchy problem for system (3.3) with initial condition

$$x(t_B) = \begin{Vmatrix} R \\ \pi \end{Vmatrix}, \quad \psi(t_B) = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} \tag{4.6}$$

satisfies conditions (4.1) and (4.2).

Since on the curve γ^+ the control is $u = 1$, it follows from (3.2) that the validity of condition (4.2) is actually equivalent to

$$\psi_2(t) < 0, \quad t \in (t_B, t_q) \tag{4.7}$$

This automatically implies condition (4.1).

Let $t_{\pi/2}$ denote the time $t = (t_q + t_B)/2$. It can be shown that at $t = t_{\pi/2}$ we have the equality $x_2(t_{\pi/2}) = \pi/2$. Hence, in view of relations (3.4) and (4.3), it follows that

$$\psi_2(t_{\pi/2}) = -1 - \frac{1}{\sqrt{1+R^2}} \tag{4.8}$$

The function $\psi_2(t)$ has no zeros in the intervals $(t_B, t_{\pi/2})$ and $(t_{\pi/2}, t_q)$.

We will prove that $\psi_2(t)$ has no zeros in the interval $(t_B, t_{\pi/2})$. Consider the function in a right half-neighbourhood of the point $t = t_B$. It follows from relations (3.3) and (4.6) that $\dot{\psi}_2(t_B) = 0$. Evaluating the second derivative

$$\ddot{\psi}_2(t) = \psi_1 u \cos x_2 - \psi_2 u \frac{\sin x_2}{x_1} \tag{4.9}$$

taking condition (4.6) into account, we obtain $\ddot{\psi}_2(t_B) = -1$, that is, the function $\psi_2(t)$ decreases monotonically in some neighbourhood of the point $t = t_B$. Suppose $\psi_2(t)$ has N zeros in the interval $(t_B, t_{\pi/2})$, where $N > 0$. Suppose the first zero is reached at a point t', $t_B < t' < t_{\pi/2}$. We then obtain from (4.3) that

$$\psi_1(t') = -\frac{1}{\cos x_2} \tag{4.10}$$

Hence, since over this time interval $x_2 \in (\pi/2, \pi)$, it follows that $\psi_1(t') > 1$. On the other hand, using Eqs (3.3), it can be shown that for $t \in (t_B, t')$ it is true that $\dot{\psi}_1(t) < 0$. Then, by condition (4.6), $\psi_1(t') < 1$. This contradiction shows that the function $\psi_2(t)$ has no zeros in the interval $(t_B, t_{\pi/2})$.

We will show that $\psi_2(t)$ has no zeros in the interval $(t_{\pi/2}, t_q)$. To prove this, we transform system (3.3) to the independent variables x_2. This may be done, since it follows from relations (3.3) and (3.4) that

$$\dot{x}_2 < 0, \quad t \in [t_B, t_q] \tag{4.11}$$

After this change of independent variable, system (3.3) becomes

$$\dot{x}_1 = \frac{x_1 \cos x_2}{x_1 - \sin x_2}$$
$$\dot{\psi}_1 = -\frac{\psi_2 \sin x_2}{x_1(x_1 - \sin x_2)}, \quad \dot{\psi}_2 = \frac{\psi_1 x_1 \sin x_2}{x_1 - \sin x_2} + \psi_2 \frac{\cos x_2}{x_1 - \sin x_2} \tag{4.12}$$

Setting $R = 0$ in (3.4), we have $x_1(x_2) = 2\sin x_2$. System (4.12) may be solved analytically in this case. Functions ψ_1^*, ψ_2^* solving the system have the form

$$\psi_1^*(x_2) = -\cos x_2, \quad \psi_2^*(x_2) = -2\sin^2 x_2 \tag{4.13}$$

Hence it follows that

$$\left|\psi_1^*(x_2)\right| \le 1, \quad x_2 \in [0, \pi] \tag{4.14}$$

We will show that the following estimate holds

$$\psi_1(x_2) \ge \psi_1^*(x_2), \quad x_2 \in [0, \pi] \tag{4.15}$$

Consider the function $\Delta(x_2) = \psi_1(x_2) - \psi_1^*(x_2)$, setting $R > 0$ in (3.4). It follows from relations (3.3), (4.6) and (4.13) that $\Delta(\pi) = 0$, $\dot{\Delta}(\pi) = 0$. After evaluating the second derivative, we obtain $\ddot{\Delta}(\pi) = 1$.

Suppose that for some x_2, the inequality $\Delta(x_2) < 0$ holds. Then a point $x_2 = x_2'$, $x_2' \in (0, \pi)$ exists, at which

$$\Delta(x_2') = 0, \quad \dot{\Delta}(x_2') > 0 \tag{4.16}$$

Let us evaluate $\dot{\Delta}(x_2')$ using system (4.12). To do this, we use equality (4.3) to express ψ_2 in terms of ψ_1 and substitute into the second equation of (4.12). Next, substituting (3.4) into the resulting equality, we find that

$$\dot{\psi}_1(x_2') = (1 + \psi_1 \cos x_2') \frac{\sin x_2'}{\sin^2 x_2' + R^2} \tag{4.17}$$

Taking into account that $\psi_1(x_2') = \psi_1^*(x_2')$, we deduce from relations (4.13) and (4.17) that

$$\dot{\Delta}(x_2') = -\frac{R^2 \sin x_2'}{\sin^2 x_2' + R^2}$$

Since $x_2' \in [0; \pi]$, it follows from this equality that $\dot{\Delta}(x_2') < 0$, contradicting the inequality in (4.16).

Suppose that the function $\psi_2(t)$ vanishes in the interval $(t_{\pi/2}, t_q)$ at $t = t''$, $t_q < t'' < t_{\pi/2}$. Then, using equality (4.3), we obtain $\psi_1(t') < -1$, which contradicts estimate (4.15).

We will now prove that in the interval $[t_0, t_N]$ the function $\psi(t)$, as a solution of the Cauchy problem for system (3.3) with boundary condition

$$x(t_N) = \begin{Vmatrix} x_N \\ \pi \end{Vmatrix}, \quad \psi(t_N) = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} \tag{4.18}$$

where x_N is the distance between the players at point N, also satisfies conditions (4.1) and (4.2). Since in the region G_3 we have $u = -1$, it follows from (3.3) that it will suffice to show that

$$\psi_2(t) > 0, \quad t \in (t_0, t_N) \tag{4.19}$$

This will automatically imply condition (4.1).

Consider the function $\psi_2(t)$ in a left half-neighbourhood of the point $t = t_N$. By relations (3.3), (4.9) and (4.18), we have

$$\psi_2(t_N) = 0, \quad \dot{\psi}_2(t_N) = 0, \quad \ddot{\psi}_2(t_N) = 1 \tag{4.20}$$

Hence it follows that the function $\psi_2(t)$ is positive in some left half-neighbourhood of the point t_N.

If $x_2(t_0) \ge \pi/2$, the technique used to prove inequality (4.19) is analogous to that used for the case $t \in (t_B, t_{\pi/2})$.

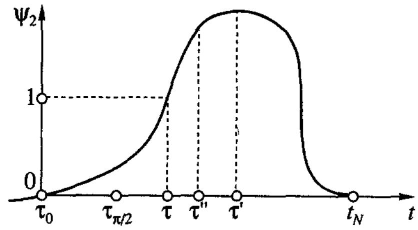

Fig. 7

Let $x_2(t_0) < \pi/2$. Let $\tau_{\pi/2}$ denote the time t at which $x_2(\tau_{\pi/2}) = \pi/2$. Using relation (4.3), it can be shown that at this point $\psi_2(\tau_{\pi/2}) > 0$.

Suppose the zero of the function $\psi_2(t)$ in the interval $(t_0, \tau_{\pi/2})$ closest to the point $\tau_{\pi/2}$ is at $t = \tau_0$, $t_0 < \tau_0 < \tau_{\pi/2}$ (Fig. 7). The, by virtue of relations (3.3), (4.3) and (4.9), we obtain

$$\psi_2(\tau_0) = 0, \quad \dot{\psi}_2(\tau_0) = \mathrm{tg}x_2, \quad \ddot{\psi}_2(\tau_0) = 1 \tag{4.21}$$

It follows from relations (4.20) and (4.21), by the continuity of the function $\ddot{\psi}_2(t)$, that it has at least two zeros in the interval (τ_0, t_N). Suppose the first zero of $\ddot{\psi}_2(t)$ is at the point $t = \tau$

$$\ddot{\psi}_2(\tau) = 0 \tag{4.22}$$

(Fig. 7). Let us find the value of $\psi_2(t)$ at $t = \tau$. By equalities (4.9) and (4.22), we have

$$\psi_1(\tau) = \psi_2(\tau)\sin x_2/(x_1 \cos x_2) \tag{4.23}$$

Substituting this expression into Eq. (4.3), we obtain

$$\psi_2(\tau) = 1 \tag{4.24}$$

Hence, by (3.3) and (4.23), it follows that

$$\dot{\psi}_2(\tau) = -1/(x_1 \cos x_2) \tag{4.25}$$

On the other hand, equalities (4.21) and (4.22) imply the estimate

$$\dot{\psi}_2(\tau) > 0 \tag{4.26}$$

By (4.25) and (4.26), we have $\tau > \tau_{\pi/2}$ and, using (4.23), we obtain

$$\psi_1(\tau) < 0 \tag{4.27}$$

It follows from (4.20), (4.24) and (4.26) that $t = \tau'$, $\tau < \tau' < t_N$ exists for which

$$\dot{\psi}_2(\tau') = 0 \tag{4.28}$$

Of all possible τ', choose that closest to the point $t = \tau$ (Fig. 7). By virtue of relations (3.3) and (4.28), we obtain

$$\psi_1(\tau') = -\psi_2 \cos x_2/(x_1 \sin x_2)$$

Consequently, $\psi_1(\tau') > 0$. It then follows from inequality (4.27), by the continuity of the function $\psi_1(t)$, that $t = \tau''$ exists (Fig. 7) such that $\psi_1(\tau'') = 0$. Hence, by relation (4.3), it follows that

$$\psi_2(\tau'') = 1/(1 + \sin x_2/x_1)$$

Then

$$\psi_2(\tau'') < 1 \tag{4.29}$$

But it follows from relations (4.26) and (4.28) that

$$\dot{\psi}_2(t) > 0, \quad t \in (\tau, \tau') \tag{4.30}$$

which, by equality (4.24), contradicts inequality (4.29).

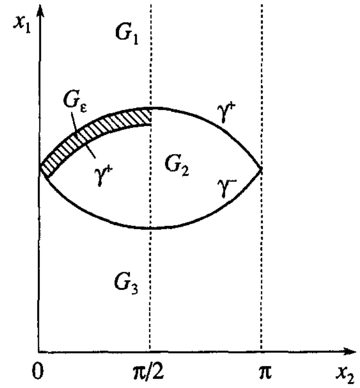

Fig. 8

The technique for defining the function $\psi(t)$ in the remaining cases of the position of q and G is analogous.

It can be shown that the synthesis constructed in the region G satisfies all the regularity conditions of [4], with the exception of the condition that the time of motion to the target point O must be continuous ("condition E"). In the terminology of [4], the sets $G_1\backslash\gamma^+$, $G_2\backslash\gamma^-$, G_3, γ^+, γ^-, SD and β are then cells of the first kind, and the points O and B are cells of the second kind. Condition E fails to hold on the arc γ^+ at $x_2 \in [0, \pi/2)$ (Fig. 5).

We now use some additional constructions in the domain G. Fix an arbitrary number $\varepsilon \in (0, R)$. Using Eq. (3.5), which defines the curve γ^+, we construct a curve γ_ε^+, replacing R in (3.5) by the quantity $R' = R - \varepsilon$. Let G_ε denote the interior of the set bounded by the curves γ^+, γ^- and γ_ε^+ and the half-line $x_2 = \pi/2$ (Fig. 8). It is not difficult to verify that all the conditions for regular synthesis are satisfied on the set $G\backslash G_\varepsilon$. By a well-known theorem (in [4, (3.19)]) and the arbitrary nature of ε, it follows that the synthesis is optimal.

5. CONCLUSION

Thus, the optimal control problem obtained as a simplification of the problem of differential game theory has been solved. This problem may now be developed in two directions: (1) using the synthesis just obtained in the initial problem as an approximate control for player I; (2) applying the method of continuation with respect to the parameter V_2 to solve more complicated optimal control problems, in particular, for the three-dimensional problem, in which it is also assumed that player II is motionless, but the orientation of his velocity vector is given.

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REFERENCES

1. ISSACS, R., *Differential Games*. Wiley, New York, 1965.
2. CHERNOUS'KO, F. L. and MELIKYAN, A. A., *Games Problems of Control and Search*. Nauka, Moscow, 1978.
3. PONTRYAGIN, L. S., BOLTYANSKII, V. G., GAMKRELIDZE, R. V. and MISHCHENKO, Ye. F., *Mathematical Theory of Optimal Processes*. Fizmatgiz, Moscow, 1961.
4. BOLTYANSKII, V. G., *Mathematical Methods of Optimal Control*. Nauka, Moscow, 1969.
5. PATSKO, V. S., PYATKO, S. G. and FEDOTOV, A. A., The three-dimensional attainability set of a non-linear control system. *Izv. Ross. Akad. Nauk. Teoriya i Sistemy Upravleniya*, 2003, 3, 8–16.
6. BERDYSHEV, Yu. I., Synthesis of optimal control for a certain 3rd-order system. In Problems of the Analysis of Non-linear Automatic Control Systems. *Trudy IMM UNTs Akad. Nauk SSSR, Sverdlovsk*, 1973, No. 12, 91–100.
7. BERDYSHEV, Yu. I., Synthesis of time-optimal control for a certain non-linear fourth-order system. *Prikl. Mat. Mekh.*, 1975, **39**, 6, 985–994.
8. ROZENBERG, G. S., Construction of trajectories of optimal pursuit. *Avtomatika i Telemekhanika*, 1965, **26**, 4, 269–633.
9. BOLYCHEVTSEV, E. M., A problem of optimal control. *Vestnik MGU*. Ser. 1. Matematika, Mekhanika, 1968, 1, 91–98.

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